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A Note on Certain Classes of Analytic Functions Defined by Salagean Operator

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Abstract. Let $\mathcal{A}$ be the class of functions of the form $f(z) = z+\sum_{m=2}^{\infty} a_m z^m$ which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. We define the new subclasses $\mathcal{M}(\alpha, \beta, n)$ and $\mathcal{M}^{\star}(\alpha, \beta, n)$ of analytic functions. The object of the present paper is to derive some convolution properties of functions f in the class $\mathcal{M}^{\star}(\alpha, \beta, n)$.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f of the form:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1.1.1}$$

defined in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Salagean [4] introduced the following operator, $D^n : \mathcal{A} \longrightarrow \mathcal{A}$ defined as: $D^0 f(z) = f(z)$, $D^1 f(z) = D(f(z)) = zf'(z)$ and $D^n f(z) = D(D^{n-1} f(z))$, for $(n \in \mathbb{N} = \{1, 2, 3, ...\})$.

We note that if f is of the form (1.1.1), then

$$D^n f(z) = z + \sum_{m=2}^{\infty} m^n \ a_m \ z^m \tag{1.1.2}$$

We define the subclass $\mathcal{M}(\alpha, \beta, n)$ as the class of all functions $f \in \mathcal{A}$ which satisfy

$$\Re\left\{\frac{z(D^n f(z))'}{D^n f(z)}\right\} < \alpha \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| + \beta, \quad (z \in \mathcal{U}) \tag{1.1.3}$$

for some $\alpha \leq 0$, $\beta > 1$ and $n \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$.
We note that $\mathcal{M}(\alpha, \beta, 0) \equiv \ \mathcal{MD}(\alpha, \beta)$, $\ \mathcal{M}(\alpha, \beta, 1) \equiv \mathcal{ND}(\alpha, \beta)$.
The classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ were studied by Owa [2].

2. Coefficient inequalities for the classes

In this section, we derive sufficient conditions for functions to be in the class $\mathcal{M}(\alpha, \beta, n)$ in terms of coefficient inequalities.

Theorem 2.1. *If $f \in \mathcal{A}$ satisfies:*

$$\sum_{m=2}^{\infty} m^n\{|m-\beta+1| + |m-\beta-1| - 2\alpha(m-1)\}|a_m| \leq \beta - |2-\beta| \tag{2.2.1}$$

for some $\alpha \leq 0, \beta > 1$ and $n \in \mathbb{N}_0$, then $f \in \mathcal{M}(\alpha, \beta, n)$.

Proof. We suppose that

$$\sum_{m=2}^{\infty} m^n\{|m-\beta+1| + |m-\beta-1| - 2\alpha(m-1)\}|a_m| \leq \beta - |2-\beta| \tag{2.2.2}$$

for $f \in \mathcal{A}$. It suffices to show that

$$\left|\frac{\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \beta\right) + 1}{\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \beta\right) - 1}\right| < 1, \quad (z \in \mathcal{U}). \tag{2.2.3}$$

We have,

$$\left|\frac{\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \beta\right) + 1}{\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \beta\right) - 1}\right|$$

$$= \left| \frac{z(D^n f(z))' - \alpha e^{i\theta} |z(D^n f(z))' - (D^n f(z))| - \beta D^n f(z) + D^n f(z)}{z(D^n f(z))' - \alpha e^{i\theta} |z(D^n f(z))' - (D^n f(z))| - \beta D^n f(z) - D^n f(z)} \right|$$

$$= \left| \frac{z + \sum_{m=2}^{\infty} m^{n+1} a_m z^m - \alpha e^{i\theta} \left| \sum_{m=2}^{\infty} m^n (m-1) a_m z^m \right| - \beta z - \beta \sum_{m=2}^{\infty} m^n a_m z^m + z + \sum_{m=2}^{\infty} m^n a_m z^m}{z + \sum_{m=2}^{\infty} m^{n+1} a_m z^m - \alpha e^{i\theta} \left| \sum_{m=2}^{\infty} m^n (m-1) a_m z^m \right| - \beta z - \beta \sum_{m=2}^{\infty} m^n a_m z^m - z - \sum_{m=2}^{\infty} m^n a_m z^m} \right|$$

$$< \frac{|2-\beta| + \sum_{m=2}^{\infty} m^n \, |m-\beta+1| \, |a_m| - \alpha \sum_{m=2}^{\infty} m^n (m-1) \, |a_m|}{\beta - \sum_{m=2}^{\infty} m^n \, |m-\beta-1| \, |a_m| + \alpha \sum_{m=2}^{\infty} m^n (m-1) \, |a_m|}.$$

The last expression is bounded above by 1 if

$$|2-\beta| + \sum_{m=2}^{\infty} m^n |m-\beta+1||a_m| - \alpha \sum_{m=2}^{\infty} m^n (m-1)|a_m|$$

$$\leq \beta - \sum_{m=2}^{\infty} m^n |m-\beta-1||a_m| + \alpha \sum_{m=2}^{\infty} m^n (m-1)|a_m| \tag{2.2.4}$$

which is equivalent to our condition

$$\sum_{m=2}^{\infty} m^n \{|m-\beta+1| + |m-\beta-1| - 2\alpha(m-1)\}|a_m| \leq \beta - |2-\beta| \tag{2.2.5}$$

of the Theorem. □

3. Relation for $\mathcal{M}^\star(\alpha, \beta, n)$

By Theorem 2.1, the class $\mathcal{M}^\star(\alpha, \beta, n)$ is considered as the subclass of $\mathcal{M}(\alpha, \beta, n)$ consisting of f satisfying:

$$\sum_{m=2}^{\infty} m^n \{|m-\beta+1| + |m-\beta-1| - 2\alpha(m-1)\}|a_m| \leq \beta - |2-\beta| \tag{3.3.1}$$

for some $\alpha \leq 0, \beta > 1$ and $n \in \mathbb{N}_0$.

We note that $\mathcal{M}^\star(\alpha, \beta, 0) \equiv \mathcal{MD}^\star(\alpha, \beta)$ and $\mathcal{M}^\star(\alpha, \beta, 1) \equiv \mathcal{ND}^\star(\alpha, \beta)$. The classes$\mathcal{MD}^\star(\alpha, \beta)$ and $\mathcal{ND}^\star(\alpha, \beta)$ were studied by Owa [2].
By the coefficient inequalities for the class $\mathcal{M}^\star(\alpha, \beta, n)$, we observe:

Theorem 3.1. *If $f \in \mathcal{A}$, then $\mathcal{M}^\star(\alpha_1, \beta, n) \subseteq \mathcal{M}^\star(\alpha_2, \beta, n)$ for some α_1, α_2 such that $\alpha_1 \leq \alpha_2 \leq 0$.*

Proof. For $\alpha_1 \leq \alpha_2 \leq 0$, we have

$$\sum_{m=2}^{\infty} m^n \{|m-\beta+1| + |m-\beta-1| - 2\alpha_2(m-1)\}|a_m|$$

$$\leq \sum_{m=2}^{\infty} m^n\{|m-\beta+1|+|m-\beta-1|-2\alpha_1(m-1)\}|a_m|. \tag{3.3.2}$$

Therefore, if $f \in \mathcal{M}^\star(\alpha_1,\beta,n)$, then $f \in \mathcal{M}^\star(\alpha_2,\beta,n)$. □

4. Convolution of the class $\mathcal{M}^\star(\alpha,\beta,n)$

For analytic function f_i given by

$$f_i(z) = z + \sum_{m=2}^{\infty} a_{m,i}\, z^m, \quad (i = 1,2,3,...,p). \tag{4.4.1}$$

The Hadamard product of $f_1, f_2, ..., f_p$ is defined by:

$$(\, f_1 * f_2 * ... * f_p)(z) = z + \sum_{m=2}^{\infty} \left(\prod_{i=1}^{p} a_{m,i} \right) z^m. \tag{4.4.2}$$

Theorem 4.1. *If $f_1 \in \mathcal{M}^\star(\alpha,\beta_1,n)$ and $f_2 \in \mathcal{M}^\star(\alpha,\beta_2,n)$ for some α $(\alpha \leq 2-\sqrt{5})$ and β_1, β_2 $(1 < \beta_1,\beta_2 \leq 2)$, then $(\, f_1 * f_2\,) \in \mathcal{M}^\star(\alpha,\beta,n)$, where*

$$\beta = 1 + \frac{(\beta_1-1)(\beta_2-1)(2-\alpha)}{(\beta_1-1)(\beta_2-1)+2^n(2-\alpha-\beta_1)(2-\alpha-\beta_2)}.$$

Proof. From(3.3.1), for $f \in \mathcal{M}^\star(\alpha,\beta,n)$ with $1 < \beta \leq 2$, we have

$$\sum_{m=2}^{\infty} m^n\{(m-\beta+1)+(m-\beta-1)-2\alpha(m-1)\}|a_m|\}$$

$$\leq \sum_{m=2}^{\infty} m^n\{(m-\beta+1)+|m-\beta-1|-2\alpha(m-1)\}|a_m| \leq 2(\beta-1).$$

That is, if $f \in \mathcal{M}^\star(\alpha,\beta,n)$, then

$$\sum_{m=2}^{\infty} \frac{m^n[m(1-\alpha)+(\alpha-\beta)]}{\beta-1}|a_m| \leq 1 \tag{4.4.3}$$

Conversely, if f satisfies:

$$\sum_{m=2}^{\infty} \frac{m^n[m(1-\alpha)+(1-\beta+\alpha)]}{\beta-1}|a_m| \leq 1 \tag{4.4.4}$$

then $f \in \mathcal{M}^\star(\alpha,\beta,n)$. From (4.4.3), if $f_1 \in \mathcal{M}^\star(\alpha,\beta_1,n)$, then

$$\sum_{m=2}^{\infty} \frac{m^n[m(1-\alpha)+(\alpha-\beta_1)]}{\beta_1-1}|a_{m,1}| \leq 1 \tag{4.4.5}$$

and also if $f_2 \in \mathcal{M}^\star(\alpha,\beta_2,n)$, then

$$\sum_{m=2}^{\infty} \frac{m^n[m(1-\alpha)+(\alpha-\beta_2)]}{\beta_2-1}|a_{m,2}| \leq 1. \tag{4.4.6}$$

Applying the Schwarz's inequality, we have the following inequality:

(4.4.7)

$$\sum_{m=2}^{\infty}\sqrt{\frac{m^{2n}\{m(1-\alpha)+(\alpha-\beta_1)\}\{m(1-\alpha)+(\alpha-\beta_2)\}}{(\beta_1-1)(\beta_2-1)}}\sqrt{|a_{m,1}||a_{m,2}|}\leq 1$$

by (4.4.5) and (4.4.6). From (4.4.4) and (4.4.7), if the following inequality

$$\sum_{m=2}^{\infty}\frac{m^n[m(1-\alpha)+(1-\beta+\alpha)]}{\beta-1}|a_{m,1}||a_{m,2}|$$

(4.4.8)

$$\leq \sum_{m=2}^{\infty}\sqrt{\frac{m^{2n}\{m(1-\alpha)+(\alpha-\beta_1)\}\{m(1-\alpha)+(\alpha-\beta_2)\}}{(\beta_1-1)(\beta_2-1)}}\sqrt{|a_{m,1}||a_{m,2}|}$$

is satisfied, then we say that $(f_1 * f_2)\in\mathcal{M}^{\star}(\alpha,\beta,n)$.
This inequality holds true if

$$\frac{m^n[m(1-\alpha)+(1-\beta+\alpha)]}{\beta-1}\sqrt{|a_{m,1}||a_{m,2}|}$$

$$\leq \sqrt{\frac{m^{2n}\{m(1-\alpha)+(\alpha-\beta_1)\}\{m(1-\alpha)+(\alpha-\beta_2)\}}{(\beta_1-1)(\beta_2-1)}} \tag{4.4.9}$$

for all $m\geq 2$. Therefore, we have

$$\frac{m^n[m(1-\alpha)+(1-\beta+\alpha)]}{\beta-1}$$

$$\leq \frac{m^{2n}\{m(1-\alpha)+(\alpha-\beta_1)\}\{m(1-\alpha)+(\alpha-\beta_2)\}}{(\beta_1-1)(\beta_2-1)} \tag{4.4.10}$$

which is equivalent to

(4.4.11)

$$\beta\geq 1+\frac{[m(1-\alpha)+\alpha](\beta_1-1)(\beta_2-1)}{(\beta_1-1)(\beta_2-1)+m^n[m(1-\alpha)+(\alpha-\beta_1)][m(1-\alpha)+(\alpha-\beta_2)]}$$

for all $m\geq 2$.
Let $R(m)$ be the right hand side of the last inequality. Further, let us define $S(m)$ by numerator of $R'(m)$. Then,

$$\begin{aligned}
S(m) &= (\beta_1-1)(\beta_2-1)[(1-\alpha)(\beta_1-1)(\beta_2-1)-m^n(1-\alpha)\{m(1-\alpha)+\alpha\}^2\\
&\quad -nm^{n-1}\{m(1-\alpha)+\alpha\}^3+nm^{n-1}\{m(1-\alpha)+\alpha\}^2\beta_2\\
&\quad +nm^{n-1}\{m(1-\alpha)+\alpha\}^2\beta_1-nm^{n-1}\{m(1-\alpha)+\alpha\}\beta_1\beta_2\\
&\quad +m^n(1-\alpha)\beta_1\beta_2]\\
&\leq (\beta_1-1)(\beta_2-1)[(1-\alpha)-m^n(1-\alpha)\{m(1-\alpha)+\alpha\}^2\\
&\quad -nm^{n-1}\{m(1-\alpha)+\alpha\}^3+4nm^{n-1}\{m(1-\alpha)+\alpha\}^2\\
&\quad -nm^{n-1}\{m(1-\alpha)+\alpha\}+4m^n(1-\alpha)]\leq 0
\end{aligned}$$

for $\alpha \leq 2-\sqrt{5}$ which shows that $R(m)$ is decreasing for $m \geq 2, \alpha \leq 2-\sqrt{5}$ and for $n \in \mathbb{N}_0$. Thus $R(2)$ is the maximum of $R(m)$ for $\alpha \leq 2-\sqrt{5}$ for $n \in \mathbb{N}_0$.

$\square$

Theorem 4.2. *If $f_i \in \mathcal{M}^\star(\alpha, \beta_i, n)$ $(i = 1, 2, 3, ..., p)$ for some $\alpha(\alpha \leq 2-\sqrt{5})$ and $\beta_i(1 < \beta_i \leq 2)$, then $(f_1 * f_2 * f_3 * ... * f_p) \in \mathcal{M}^\star(\alpha, \beta, n)$, where*

$$\beta = 1 + \frac{A_p}{B_p - C_p D_p + E_p}, \quad (p \geq 2), \tag{4.4.12}$$

$$A_p = \prod_{i=1}^{p} (\beta_i - 1)(2-\alpha)^{p-1}, \tag{4.4.13}$$

$$B_p = (2-\alpha)^{p-2} \prod_{i=1}^{p} (\beta_i - 1), \tag{4.4.14}$$

$$C_p = \sum_{q=1}^{p-2} (2^n)^q \ (2-\alpha)^{p-q-2} \ (1-\alpha)^{q-1}, \tag{4.4.15}$$

$$D_p = \prod_{i=1}^{p-q} (\beta_i - 1) \prod_{L=p-q+1}^{p} (2-\alpha-\beta_L), \tag{4.4.16}$$

$$\textit{and} \quad E_p = (2^n)^{p-1} \ (1-\alpha)^{p-2} \prod_{i=1}^{p} (2-\alpha-\beta_i). \tag{4.4.17}$$

Proof. When $p = 2$, we have

$$\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2-\alpha)}{(\beta_1 - 1)(\beta_2 - 1) + 2^n(2-\alpha-\beta_1)(2-\alpha-\beta_2)} \tag{4.4.18}$$

Let us suppose that $(f_1 * f_2 * f_3 * ... * f_k) \in \mathcal{M}^\star(\alpha, \beta_0, n)$ and $f_{k+1} \in \mathcal{M}^\star(\alpha, \beta_{k+1}, n)$ where

$$\beta_0 = 1 + \frac{A_k}{B_k - C_k D_k + E_k} \quad (k \geq 2).$$

Using Theorem 4.1 and replacing β_1 by β_0 and β_2 by β_{k+1}, we see that

$$
\begin{aligned}
\beta &= 1+\frac{(\beta_0-1)(\beta_{k+1}-1)(2-\alpha)}{(\beta_0-1)(\beta_{k+1}-1)+2^n(2-\alpha-\beta_0)(2-\alpha-\beta_{k+1})}\\
&= 1+\frac{A_{k+1}}{B_{k+1}-\{2^n(2-\alpha-\beta_{k+1})B_k+2^n(1-\alpha)(2-\alpha-\beta_{k+1})\mathcal{C}_kD_k\}+E_{k+1}},\\
&= 1+\frac{A_{k+1}}{B_{k+1}-\{2^n(2-\alpha-\beta_{k+1})B_k+C_k^+D_{k+1}\}+E_{k+1}}\\
&= 1+\frac{A_{k+1}}{B_{k+1}-C_{k+1}D_{k+1}+E_{k+1}}
\end{aligned}
$$

where

$$C_k^+=\sum_{q=2}^{k-1}(2^n)^q\ (2-\alpha)^{k-q-1}\ (1-\alpha)^{q-1}.$$

$\square$

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